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q bosons as quasifermions

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The thermodynamic properties of both free and weakly interacting gases of independent q bosons ($A_p^- A_p^+ - qA_p^+ A_p^- = 1$, $[A_p^\mp, A_k^\mp] = 0$, $q > 1$) with a gap in the frequency spectrum are considered. Even for q slightly exceeding 1, the q Bose gas behaves as a Fermi gas rather than a Bose gas: for small values of the temperature the distribution function consists of a finite number of the Fermi "steps." The main thermodynamic characteristics (pressure, specific heat, etc.) are evaluated for three- and two-dimensional cases. New physical effects such as "stepped" behavior of the specific heat of a two-dimensional q Bose gas, phase separation of the ground state of the interacting gas, and an increase of the effective mass of quasifermion states at large concentrations are predicted.

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I. INTRODUCTION

Although many objects constructed from q bosons and q oscillators were intensively studied from both mathematical [1] and physical [2] points of view, the search for real physical systems with q bosons remains of great interest up to now. In this respect, it is interesting to investigate statistical properties of q bosons to compare them with those of various systems.

In earlier papers [3,4], statistical properties of noninteracting q bosons were considered, whereas the q-deformed Einstein-Planck distribution was studied in [5]. Following this direction we will study the behavior of both a noninteracting and an interacting q Bose gas with a gap in the frequency spectrum. On the basis of the canonical statistical distribution we will find the main thermodynamical properties which appear to be drastically different from those of ordinary bosons.

The present paper deals with q bosons considered as particles with corresponding operators: annihilation A_p^- and creation A_p^+ (p denotes the mode label), defined (together with the number operators N_p) using the commutation relations [6,7]:

$$[A_k^-, A_p^+] = \delta_{kp}[1+(q-1)A_p^+ A_p^-] , \quad q > 1 \qquad (1.1a)$$

$$[N_k, A_p^\mp] = \mp \delta_{kp} A_p^\mp , \qquad (1.1b)$$

where the number operator N_p can be expressed through $A_p^\mp$ as

$$N_p = \log_q[1+(q-1)A_p^+ A_p^-] \qquad (1.1c)$$

(mainly in the present paper we will consider the case $q-1 \ll 1$). When $q \to 1$, (1.1) becomes the usual oscillator algebra. The operator of the total number of particles

$$N = \sum_p N_p \qquad (1.2)$$

commutes with the Hamiltonian of noninteracting q bosons

$$H_0 = \sum_p \Omega_p A_p^+ A_p^- . \qquad (1.3)$$

In what follows we will assume that the frequency spectrum has the gap Ω_0

$$\Omega_p \geq \Omega_0 > 0 . \qquad (1.4)$$

As will be shown below, such a choice of frequency spectrum leads to nontrivial physical consequences.

It follows from (1.1c) that the Hamiltonian (1.3) is a nonlinear function of N_p

$$H_0 = \sum_p \Omega_p [N_p]_q , \quad \text{where } [x]_q \equiv \frac{q^x - 1}{q-1} , \qquad (1.5)$$

where, in the limiting case, $q \to 1$ turns into the one for the usual bosons

$$H_0 = \sum_p \Omega_p N_p .$$

Our choice of $q > 1$ is stipulated by the fact that for $|q| < 1$ the energy spectrum of the q oscillator contains a continuous part [8,9] and the number operator cannot be correctly defined in this case.

II. STATISTICS OF NONINTERACTING q BOSONS

To study the thermodynamic properties of systems described by the Hamiltonian (1.3) we have to average corresponding values with the canonical distribution

$$w = Z^{-1}\exp[(\mu N - H_0)/T]$$
$$= Z^{-1} \prod_p \exp[(\mu N_p - \Omega_p A_p^+ A_p^-)/T] ,$$

where μ is the chemical potential, and $Z = \mathrm{Tr}\exp[(\mu N - H_0)/T]$.

To start with, we calculate the mean number of q bosons (mean occupation number) in the pth mode,

$$n_p \equiv \langle N_p \rangle = Z^{-1}\mathrm{Tr}\{N_p \exp[(\mu N - H_0)/T]\}$$
$$= \sum_{L=0}^{\infty} L \exp(\{\mu L - \Omega_p [L]_q\}/T) \Big/ \sum_{L=0}^{\infty} \exp(\{\mu L - \Omega_p [L]_q\}/T) \tag{2.1}$$

[the expression for the Hamiltonian in terms of N_p (1.5) has been used].

Even though n_p cannot be presented in explicit form, its asymptotic expression for low temperatures

$$T \ll \mu(q-1) \tag{2.2}$$

can be obtained.

In the limit $T \to 0$ it is easy to find the occupation number in the ground state,

$$n_p = 0 \quad \text{if } \Omega_p > \mu \;,$$
$$n_p = 1 \quad \text{if } \mu q^{-1} < \Omega_p < \mu \;,$$
$$\cdots$$
$$n_p = L \quad \text{if } \mu q^{-L} < \Omega_p < \mu q^{-L+1} \;.$$

This distribution can be rewritten as

$$n_p = f(\Omega_p;\mu) \equiv [1 + \log_q(\mu/\Omega_p)]\theta(\mu - \Omega_p) \;, \tag{2.3}$$

where the square brackets denote the integer part and

$$\theta(x) \equiv \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \;. \end{cases}$$

The distribution (2.3) can be easily obtained by direct minimization of the sum (1.5) under the condition $N = \text{const}$.

From (1.5) and (2.3) we obtain that n_p changes in the ranges

$$0 \le n_p \le f(\Omega_0;\mu) \;.$$

The expression in (2.3) may be interpreted as a "multistepped" analog of the Fermi-Dirac distribution at $T = 0$ (see Fig. 1), which appears due to the gap in the frequency spectrum and the nonlinearity of the Hamiltonian. Nevertheless in the limiting case $q \to 1$ we will return to the ordinary Bose-Einstein distribution at $T = 0$, when all the bosons are collected in the mode with $\Omega = \Omega_0$ (see the next section). The same result takes place if $\Omega_0 \to 0$.

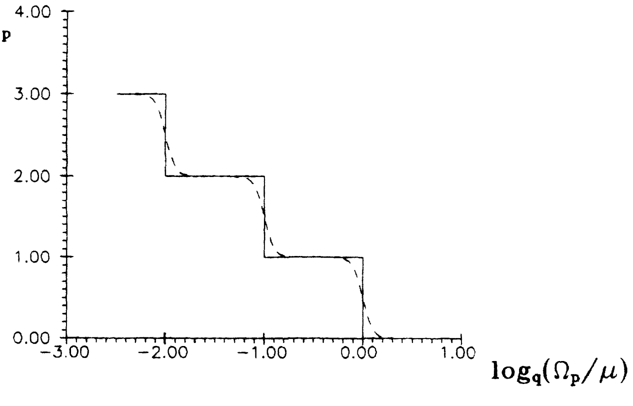

FIG. 1. The distribution functions for $T = 0$ and $T_1 = 0.05\mu(q-1)$ (dashed line). The distribution for T_1 has been obtained by numerical calculations ($q = 1.01$) from Eq. (2.1).

The mean occupation number at low temperatures (2.2) is obtained by an approximation of (2.1) in the neighborhood of the frontiers of "steps," i.e., when

$$\Omega_p = \mu q^{-M} + \delta \;.$$

Replacing L in (2.1) with $L = N + M$ we easily obtain

$$n_p = M + \frac{\sum_{N=-M}^{\infty} N \exp(\{\mu N - (\mu + \delta q^M)[N]_q\}/T)}{\sum_{N=-M}^{\infty} \exp(\{\mu N - (\mu + \delta q^M)[N]_q\}/T)} \;. \tag{2.4}$$

First, we estimate the denominator of this equation

$$\sum_{N=-M}^{\infty} \exp(\{\mu N - (\mu + \delta q^M)[N]_q\}/T)$$
$$= \sum_{N=-M}^{-1} \cdots + 1 + e^{-\delta q^M/T} + \sum_{N=2}^{\infty} \cdots \;.$$

We readily estimate the two remaining sums using the series expansion of $[N]_q$,

$$N - [N]_q = -\frac{N(N-1)}{2}(q-1) + \cdots \;; \quad q - 1 \ll 1$$

and we find

$$\sum_{N=-M}^{-1} \cdots + \sum_{N=2}^{\infty} \cdots = O(e^{-\mu(q-1)/T}) \;.$$

Therefore

$$\sum_{N=-M}^{\infty} \exp(\{\mu N - (\mu + \delta q^M)[N]_q\}/T)$$
$$= 1 + e^{-\delta q^M/T} + O(e^{-\mu(q-1)/T}) \;.$$

The numerator of Eq. (2.4) can be estimated in an analogous way:

$$\sum_{N=-M}^{\infty} N \exp(\{\mu N - (\mu + \delta q^M)[N]_q\}/T)$$
$$= e^{-\delta q^M/T} + O(e^{-\mu(q-1)/T}) \;.$$

Thus, due to (2.2), one obtains the expansion of n_p,

$$n_p = M + \frac{e^{-\delta q^M/T} + O(e^{-\mu(q-1)/T})}{1 + e^{-\delta q^M/T} + O(e^{-\mu(q-1)/T})}$$
$$= M + \frac{1}{e^{\delta q^M/T} + 1} + \cdots \;.$$

Combining these expansions for $0 \le M < \infty$ we get the final formula for n_p,

$$n_p=\sum_{L=0}^{\infty}\{\exp[(\Omega_p q^L-\mu)/T]+1\}^{-1}\,, \tag{2.5}$$

which reduces to (2.3) when $T=0$.

It is worth mentioning that in the situation when

$$\mu<q\Omega_0\,, \tag{2.6}$$

$f(\Omega_0;\mu)=1$ $(\Omega_0<\Omega_p)$, the distribution in (2.5) [of course at temperatures (2.2) when all terms with $L>0$ can be neglected in (2.5)] becomes a standard Fermi-Dirac distribution:

$$n_p=\{\exp[(\Omega_p-\mu)/T]+1\}^{-1}\,. \tag{2.7}$$

Although the q bosons do not obey the Pauli principle, and any number of them could be in any mode, energetic considerations lead to the distribution (2.7) [for a sufficiently large value of Ω_0, see (2.6)], and consequently in this case q bosons become thermodynamically nondistinguishable from ordinary fermions.

To describe the behavior of a q Bose gas, the value of the total energy is needed,

$$E=Z^{-1}\mathrm{Tr}\{H_0\exp[(\mu N-H_0)/T]\}=\sum_p \varepsilon_p\,, \tag{2.8}$$

where

$$\varepsilon_p\equiv\langle E_p\rangle=\sum_{L=0}^{\infty}\Omega_p[L]_q\exp\{(\mu L-\Omega_p[L]_q)/T\}\Big/\sum_{L=0}^{\infty}\exp\{(\mu L-\Omega_p[L]_q)/T\}\,.$$

Similar to (2.1), the expressions in (2.8) can be evaluated at low temperatures (2.2) in a similar manner,

$$\varepsilon_p=\sum_{L=0}^{\infty}\Omega_p q^L[\exp[(\Omega_p q^L-\mu)/T)]^{-1}\,. \tag{2.9}$$

One can also find an exact expression for the energy ε_p in the ground state as a limiting case of (2.9) at $T=0$

$$\varepsilon_p=E_p=\Omega_p[f(\Omega_p;\mu)]_q\,,$$

where the function $f(\Omega_p;\mu)$ has been defined in (2.3).

Now we are ready to write the final expressions for the total number of particles N and the total energy E. Introducing the density of states $g(\Omega)$ as a number of modes with a frequency between Ω and $\Omega+d\Omega$, we replace the summations (2.8) and (1.2) with integrations as

$$\begin{aligned}N&=\int_{\Omega_0}^{\infty}n(\Omega)g(\Omega)d\Omega\\&=\sum_{L=0}^{L_{\max}}\left\{G(\mu q^{-L})+\frac{\pi^2T^2}{6}q^{-2L}\left[\frac{dg}{d\Omega}\right]_{\Omega=\mu q^{-L}}+\cdots\right\}\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}E&=\int_{\Omega_0}^{\infty}\varepsilon(\Omega)g(\Omega)d\Omega\\&=\sum_{L=0}^{L_{\max}}\left\{q^LF(\mu q^{-L})\right.\\&\qquad\left.+\frac{\pi^2T^2}{6}q^{-L}\left[\frac{d\Omega g}{d\Omega}\right]_{\Omega=\mu q^{-L}}+\cdots\right\}\end{aligned} \tag{2.11}$$

where

$$G(\Omega)=\int_{\Omega_0}^{\Omega}g(x)dx\,,\quad F(\Omega)=\int_{\Omega_0}^{\Omega}xg(x)dx\,,$$

and (2.12)

$$L_{\max}=f(\Omega_0;\mu)-1\,.$$

It follows from (2.10) that the chemical potential at zero temperature μ_0 is a solution of the equation

$$N=\sum_{L=0}^{L_{\max}}G(\mu_0q^{-L})\,, \tag{2.13}$$

and the energy of the ground state is immediately obtained from (2.11),

$$E_0=\sum_{L=0}^{L_{\max}}q^LF(\mu_0q^{-L})\,. \tag{2.14}$$

The value of the chemical potential at low temperatures (2.2) can be calculated by its expansion in the neighborhood of $T=0$:

$$\mu(T)=\mu_0-\frac{\pi^2T^2}{6}\frac{\partial}{\partial\mu_0}\left[\ln\sum_{L=0}^{L_{\max}}q^{-L}g(\mu_0q^{-L})\right]+\cdots\,. \tag{2.15}$$

Finally, inserting (2.15) into the expression for the total energy (2.11), one gets the specific heat $C_V\equiv(\partial E/\partial T)_V$ at a constant volume

$$C_V=\frac{\pi^2T}{3}\sum_{L=0}^{L_{\max}}q^{-L}g(\mu_0q^{-L})\,, \tag{2.16}$$

and we see that, similar to the ordinary fermions, the specific heat of a free q Bose gas is proportional to the temperature [if the condition (2.2) takes place].

Similarity between the properties of a q Bose and a Fermi gas becomes especially apparent in the situation (2.6), or as follows from (2.13) when

$$N<G(q\Omega_0)=\int_{\Omega_0}^{q\Omega_0}g(x)dx\,. \tag{2.17}$$

In this case, the formula (2.16) reduces to that for ordinary fermions [10].

III. "QUASIFERMION" STATES

In the beginning of this section we consider a free isotropic q deformed gas bounded by the volume V of a three-dimensional isotropic space. On the basis of the results obtained in the previous section we will deduce the equation of state at the absolute zero temperature and find the specific heat at low temperatures, restricting our study by the physically relevant case of a small value of the deformation parameter

$$\omega \equiv \ln q \ll 1 \ . \tag{3.1}$$

Let us consider the gas with p and m being the momentum and mass of q bosons. Taking its dispersion law in the form

$$\Omega_p = \Omega_0 + \frac{p^2}{2m} \tag{3.2}$$

from standard considerations in statistical mechanics [10], we find the density of states as

$$g(\Omega) = \frac{1}{(2\pi\hbar)^3}\frac{d\Gamma}{d\Omega} = \frac{V}{2\pi^2\hbar^3}[2m^3(\Omega - \Omega_0)]^{1/2} \tag{3.3}$$

(here $d\Gamma = 4\pi V p^2 dp$ is the volume in phase space).

Substitution of (3.3) into (2.13) yields the relation for the concentration $\rho_V \equiv N/V$ (number of bosons in the unit volume of space)

$$\rho_V = \frac{1}{6\pi^2\hbar^3}\sum_{L=0}^{L_{\max}}[2m(\mu q^{-L} - \Omega_0)]^{3/2} \tag{3.4}$$

where $L_{\max}$ is defined in (2.12).

To start with, consider the case

$$f(\Omega_0;\mu) \gg 1 \ . \tag{3.5}$$

The summation in (3.4) then can be replaced by the integration due to the small value of the deformation parameter (3.1). In such an approximation

$$\rho_V = \frac{(2m\Omega_0)^{3/2}}{3\pi^2\hbar^3\omega}\left[\frac{z^3}{3} - z + \arctan(z)\right] , \tag{3.6}$$

where the new parameter z is

$$z = [(\mu - \Omega_0)/\Omega_0]^{1/2} \ .$$

The energy of the system at $T=0$ can be expressed in terms of z from (2.14) in a similar manner,

$$E = \frac{(2m\Omega_0)^{5/2}}{30\pi^2\hbar^3 m\omega}Vz^5 \ . \tag{3.7}$$

From (2.16) one obtains specific heat at low temperatures

$$C_V = \frac{(2m\Omega_0)^{3/2}}{18\hbar^3\omega\Omega_0}VT\frac{z^3}{z^2+1} \ . \tag{3.8}$$

It is necessary to find z from the transcendental Eq. (3.6) in order to express energy through concentration. Although Eq. (3.6) is not exactly solvable, there are three important cases: case (a),

$$z < \omega^{1/2} , \tag{3.9}$$

case (b)

$$\omega^{1/2} \ll z \ll 1 ,$$

case (c)

$$z \gg 1 ,$$

when asymptotic [and in case (a) exact] expressions for energy and pressure $P \equiv -(\partial E/\partial V)$ can be obtained in explicit form. In case (a) the calculations can be carried out exactly without approximation (3.5).

Consider these cases separately:

Case (a): $z < \omega^{1/2}$. This case corresponds to "small" concentrations of q bosons

$$\rho_V < \frac{(2m\Omega_0\omega)^{3/2}}{15\pi^2\hbar^3} \ . \tag{3.10}$$

One can easily verify that this condition [or condition (3.9)] is equivalent to (2.17) or (2.6) and hence in this case the q Bose gas is thermodynamically nondistinguishable from a Fermi gas (these states of q Bose gas can be called "quasifermion" ones).

Case (b): $\omega^{1/2} \ll z \ll 1$. This case corresponds to "intermediate" concentrations,

$$\frac{(2m\Omega_0\omega)^{3/2}}{15\pi^2\hbar^3} \ll \rho_V \ll \frac{(2m\Omega_0)^{3/2}}{15\pi^2\hbar^3\omega} \ . \tag{3.11}$$

From (3.6) and (3.7) we get the energy of the ground state

$$E = N\Omega_0\left[1 + \frac{5z^2}{7} + \cdots\right]$$
$$= N\Omega_0\left[1 + \frac{5}{7}\left[\frac{15\pi^2\hbar^3\omega}{(2m\Omega_0)^{3/2}}\right]^{2/5}\rho_V^{2/5} + \cdots\right] \tag{3.12}$$

and the equation of state at $T=0$

$$P = \tfrac{2}{7}\Omega_0\left[\frac{15\pi^2\hbar^3\omega}{(2m\Omega_0)^{3/2}}\right]^{2/5}\rho_V^{7/5} \ . \tag{3.13}$$

One can see that in the limiting case $\omega = 0$ ($q=1$) the condition (3.11) turns into

$$0 < \rho_V < \infty ,$$

and the expressions (3.12) and (3.13) reduce to ones for the ordinary Bose gas.

Case (c): $z \gg 1$. This case corresponds to "large" concentrations

$$\rho_V \gg \frac{(2m\Omega_\rho)^{3/2}}{15\pi^2\hbar^3\omega} \ . \tag{3.14}$$

For energy, pressure, and specific heat we have the expressions

$$E = \frac{3N\hbar^2}{10m^*}(3\pi^2\rho_V)^{2/3} , \quad P = \frac{2E}{3V} ,$$
$$C_V = \frac{m^*N}{\hbar^2}T\left[\frac{\pi}{3\rho_V}\right]^{2/3}$$

which coincide with those for fermions with the effective mass

$$m^* = \frac{m}{(3\omega)^{2/3}} \tag{3.15}$$

and one can say that apart from case (a) the second type of "quasifermion" ("heavy fermion") states occurs at "large" concentrations of q bosons. Note that an analogous property takes place not only in the three-dimensional space, but also in the space of arbitrary dimensions.

Thus substantial metamorphosis has occurred when concentrations of q bosons change from 0 to ∞. Therefore q bosons behave like fermions if the concentration does not exceed some characteristic value [see (3.10)]. In an intermediate region of ρ_V the properties of a q Bose gas are different from both the usual Bose and Fermi ones, and when $\rho_V \rightarrow \infty$, the gas again displays quasifermion properties, but the effective mass m^* of the new quasifermions (heavy fermions) [see (3.15)] is different from that of the q bosons. These regions can be roughly illustrated by Fig. 2.

It is interesting to consider a two-dimensional gas with the dispersion law (3.2) when the density of states is

$$g(\Omega)=\frac{1}{(2\pi\hbar)^2}\frac{d\Gamma}{d\Omega}=\frac{mS}{2\pi\hbar^2}$$

(here S denotes two-dimensional volume) and it does not depend on the frequency. Then from (2.16) we have the following expression for the specific heat:

$$C=C_0[f(\Omega_0;\mu)]_q \tag{3.16}$$

with

$$C_0=\frac{\pi m S T}{6\hbar^2} .$$

It is seen from (3.16) and the definition of $f(\Omega_0;\mu)$ (2.3) that specific heat exhibits jumps given by

$$\Delta C_L = C_0 q^{-L}$$

when

$$\mu=\mu^{(L)}=\Omega_0 q^L , \quad L=1,2,3,\ldots .$$

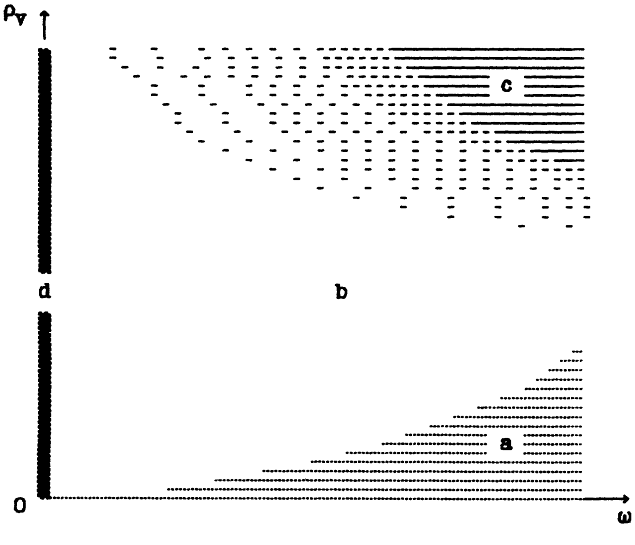

FIG. 2. Phase diagram of the q Bose gas at $T=0$. a, quasifermions; b, intermediate state; c, "heavy quasifermions;" and d, ordinary Bose condensate.

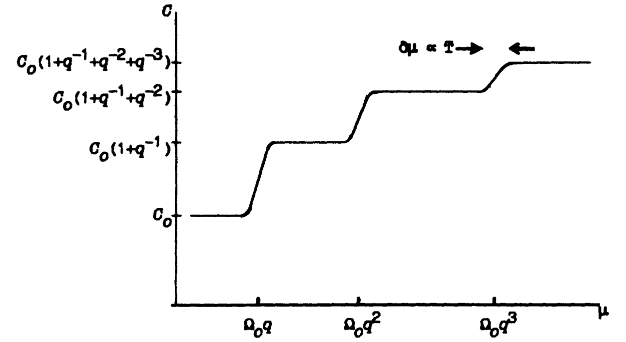

FIG. 3. Specific heat C of two-dimensional q Bose gas at low temperatures.

The concentration ρ_V is expressed in explicit form in the two-dimensional situation. So we obtain from (2.13) that the jumps of specific heat occur when

$$\rho_S=\rho_S^{(L)}=\frac{m\Omega_0}{2\pi\hbar^2}\{[L+1]_q-L-1\}$$

and the specific heat does not depend on concentration in the intervals between $\rho_S^{(L)}$ and $\rho_S^{(L+1)}$ (or $\mu^{(L)}$ and $\mu^{(L+1)}$), i.e., C is a "stepped" function of ρ_S (see Fig. 3).

Strictly speaking, such a discontinuous character (3.16) of C is due to the approximation (2.16). In fact, C is a continuous function of the chemical potential μ (or concentration ρ_S), but the approximation (3.16) is valid owing to the fact that at low temperatures (2.2) jumps of the specific heat (transition from the Lth "step" to the $L+1$th one) take place in the very small intervals of the chemical potential $\mu^{(L)}\pm\delta\mu$ (see Fig. 3), so that

$$\frac{\delta\mu}{\mu^{(L+1)}-\mu^{(L)}} \propto \frac{T}{\mu(q-1)} \ll 1 .$$

In other words, the width of the "steps" $\mu^{(L+1)}-\mu^{(L)}$ (or $\rho_S^{(L+1)}-\rho_S^{(L)}$) is much more than the transition interval $\delta\mu \propto T$ (or $\delta\rho_S^{(L)}$, respectively). It is worth mentioning that such "stepped" behavior of thermodynamic properties is a specific feature of two-dimensional fermion systems, recall, e.g., the quantum Hall effect [11].

IV. INTERACTING q BOSE GAS

Although the description of an interacting q Bose gas is a rather difficult problem, we try to obtain a qualitative picture of the behavior of the system at $T=0$ (ground state) in the frame of the model considered in the preceding section. We start with the situation when the concentration of q bosons lies in the interval (3.11) ("intermediate" concentrations). Returning to the noninteracting q Bose gas, let us rewrite expression (3.13) in the form

$$P=\rho_V T_{\text{eff}} , \tag{4.1}$$

where the meaning of the "effective temperature"

$$T_{\text{eff}}=\tfrac{2}{7}\Omega_0\left[\frac{15\pi^2\hbar^3\omega}{(2m\Omega_0)^{3/2}}\right]^{2/5}\rho_V^{2/5} \tag{4.2}$$

is clarified if one calculates the mean kinetic energy per particle,

$$\langle P^2/2m\rangle=\text{const}\,T_{\text{eff}} . \tag{4.3}$$

In other words, T_{eff} is a measure of the mean kinetic energy of the q bosons in the ground state. Thus, the relations (4.1) and (4.3) have the form that coincides with that for the usual theory of an ideal gas with temperature T_{eff}. Using this fact and supposing that there is a weak attraction between q bosons, we can consider an interacting q Bose gas in the ground state as a van der Waals gas at temperature T_{eff}, and consequently for such a gas

$$P=\frac{T_{\text{eff}}}{v-b}-\frac{a}{v^2} , \tag{4.4}$$

where $v=\rho_V^{-1}$ plays the role of the volume per fixed particle, and a,b are the van der Waals constants [for noninteracting gas $a=b=0$, we return to (4.1)].

A similar result can be deduced from perturbation theory. Indeed, choosing the Hamiltonian in the form

$$H=H_0+W_{\text{int}}$$

with the interacting part

$$W_{\text{int}}=-\frac{1}{2V}\sum_{k,p,s}W_{k,p,s}A_{p-k}^{+}A_{s+k}^{+}A_p^{-}A_s^{-} ,$$

we obtain the first-order approximation for the energy of the ground state

$$\begin{aligned}E&=E_0+\langle 0|W_{\text{int}}|0\rangle\\&=E_0-\frac{1}{2V}\sum_{p,s}(W_0+W_{p-s,p,s})\langle 0|A_p^{+}A_s^{+}A_p^{-}A_s^{-}|0\rangle\end{aligned} \tag{4.5}$$

(here $|0\rangle$ is the ground state of the noninteracting Hamiltonian and $W_0\equiv W_{0,0,0}$).

The perturbation theory is valid if the radius of interaction r_0 is much smaller than the mean distance between particles

$$r_0 \ll \rho_V^{-1/3} . \tag{4.6}$$

Remembering that from the van der Waals theory [10] it follows that $b\propto r_0^3$, the condition (4.6) can be rewritten in the form

$$v\gg b . \tag{4.7}$$

The radius of interaction in momentum space

$$p_0\propto r_0^{-1}\gg\rho_V^{1/3}$$

is much more than the maximal value of momentum of noninteracting q bosons at the same concentration, and therefore we may replace $W_{p-s,p,s}$ in (4.5) with W_0. Then, from (4.5) with due regard to (3.1), (3.5), (3.6), and (1.1c), we obtain the formula for the pressure in this approach,

$$P=\frac{T_{\text{eff}}}{v}-\frac{W_0}{v^2} \tag{4.8}$$

from which we see that for $v\gg b$ expression (4.4) practically (within this approach) coincides with (4.8) and

$$a=W_0 .$$

Now, substituting (4.2) into (4.4) we obtain the final expression for pressure,

$$P(x)=\frac{W_0}{b^2}\left[\frac{R}{x^{2/5}(x-1)}-\frac{1}{x^2}\right] ,$$

where $x=v/b$, $1<x<\infty$, and

$$R=\frac{2\Omega_0 b}{7W_0}\left[\frac{15\pi^2\hbar^3\omega}{(2m\Omega_0)^{3/2}}\right]^{2/5} .$$

The function $P(x)$ behaves like a van der Waals isotherm. So if the value of R does not exceed the critical one, $R_c\cong 0.365$, then $P(x)$ has two extrema, and according to the theory of phase transitions [10], in some interval $x_1(R)<x<x_2(R)$ phase separation takes place. Thus due to the interplay between attraction by interacting and repulsion by "nonbosonic" character of particles we obtain phase separation in the ground state of the system. The coexistence region x_2-x_1 diminishes if the deformation parameter ω and/or the interaction parameter W_0 change so that $R\to R_c$, and goes to zero at the critical point R_c.

Note that for some values of ω and W_0 (4.8) has maxima when the condition of applicability of perturbation theory (4.7) is carried out. Therefore instability $(\partial P/\partial V)>0$ appears for some concentrations already in the frame of the expression (4.8) and phase separation takes place even in this rough (without van der Waals approximation) approach.

Finally, in the situation when condition (3.10) takes place and the noninteracting q bosons have occupation number and energy distributions that coincide with those for fermions, even small attraction with a large radius

$$r_0\gg\rho_V^{-1/3} \tag{4.9}$$

leads to substantial changes of the spectrum and perturbation theory in its usual form becomes inappropriate.

V. CONCLUSIONS

It was the main purpose of this paper to study thermodynamics and statistics of a q Bose gas with a gap in the frequency spectrum. Calculations of statistical properties have resulted in expressions for the chemical potential, specific heat at low temperatures, and an equation of state at $T=0$. It was found that these properties are quite different from those for both a Bose gas and a q Bose one without a gap in the frequency spectrum. Moreover in some situations the thermodynamic properties of such a gas are quite similar to those of a Fermi gas than to a Bose gas, and as a consequence such a gas does not undergo Bose concentration. This fact seems to be unexpected because in earlier considerations q Bose condensation was demonstrated [3,4]. Therefore we note the two crucial points of our approach leading to this conclusion: (i) $q>1$ and (ii) the existence of the gap in the frequency spectrum of q bosons. It is interesting to note that in our case $\mu>\Omega_0$ in contrast to the theory of the ordinary Bose gas [10], where only the condition $\mu<\Omega_0$ is valid.

Indeed, it has already been mentioned that if $\mu<q\Omega_0$ $(q>1)$, then (2.7) takes place and q bosons behave as true fermions. But according to the theory of fermion systems [10] the condition $\mu>\Omega_0$ is carried out at small values of temperature (and hence Bose condensation does not occur if $\Omega_0>0$). Such an unusual condition is due to a strong convergence of the partition function Z (2.1) even for $q-1\ll 1$, whereas in the classical $(q=1)$ case the function Z diverges if $\mu>\Omega_0$. However, in the case $\Omega_0=0$ the equation for the zero momentum state occupation number n_0 $(p=0)$ [see (2.1)] coincides with that for ordinary bosons and this state can be occupied by the macroscopically large amount of q bosons. Thus, in contrast to our $(\Omega_0>0)$ case, q bosons without a gap $(\Omega_0=0)$ can Bose condense even in two dimensions [4] due to strong convergence of the partition function with $p\neq 0$ (i.e., $\Omega_p>0$).

In a two-dimensional situation the specific heat is a "stepped" function of concentrations (specific heat of a two-dimensional q Bose gas has an infinite number of exponentially decreasing jumps when the concentration changes from 0 to ∞).

An attempt to describe the interacting q Bose gas with a small deformation parameter has also been made in the present paper. Having considered a q Bose gas with a weak interaction in the volume V of three-dimensional space, we have obtained that phase separation may occur in the ground state of the system when the deformation parameter and the interaction parameters satisfy some specific conditions.

Note that conclusions of this paper [except for case (c) in Sec. III] are valid for all the deformations of the basic commutation relation

$$[A^-,A^+]=F(A^+A^-)\ ,\quad F(x)=1+\omega x+\cdots\ ,\quad \omega>0$$

because of small deformations (3.1) and relatively small values of occupation numbers (1.1c)

$$\omega N_p \ll 1\ ,\quad F(x)\cong 1+\omega x\ .$$

Our results can be generalized to the situation when the deformation parameter depends on the mode label, i.e., $q=q(p)$ [or $\omega=\omega(p)$].

In conclusion, one can say that when the concentration of q bosons is small enough (3.10) and the free q Bose gas has occupation numbers and energy distributions that coincide with those for a Fermi gas, this situation might be of interest in condensed matter [e.g., the hypothetical q bosons with the dispersion law for an electron $(\Omega_0=mc^2\cong 1$ MeV) behave as true fermions even for sufficiently small values of the deformation parameter such as $\omega=\ln q\cong 10^{-5}$; so the main statements of solid state physics remain the same in spite of replacing electrons with such q bosons]. In such a situation the attraction with the large radius (4.3) leads to substantial changes in the spectrum of excitations of the system and by analogy with the Fermi gas the question of whether a gap is present or not in the spectrum of excitations of q Bose gas appears to be crucial in this situation.

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